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Second Variational Formulae for Dimension Spectra

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Consider basic set for Axiom A diffeomorphism on compact surface. We derive second variational formulae for the dimension spectra of equilibrium state on the basic set with respect to the perturbations of both the potential and the dynamical system. In particular we obtain a second variational formula for the Hausdorff dimension of the basic set. These results will find their use in the study of a quadratic extremal problem for multifrcatal analysis.

KEY WORDS: Topological pressure; equilibrium states; thermodynamic formalism; hyperbolic dynamical systems.

1. INTRODUCTION

Several first derivative formulae for topological entropy, measure theoretic entropy and Hasudorff dimension are studied when certain hyperbolic dynamical system is smoothly perturbed.[7,8] In this note, we derive second derivative formulae for the dimension spectra, with the aid of the thermodynamic formalism, when both the dynamical system and the measure are perturbed. In particular, we derive an explicit second variational formula for the Hausdorff dimension of basic set and that is given in terms of the measure theoretic entropy and the correlation function (Theorem 1.1). This is a first step towards a quadratic variational problem for multifractal formalism.

Let M be a compact Riemannian surface and $\mathrm{Diff}^r(M)$ the space of C^r diffeomorphisms of M endowed with the C^r topology. Consider a parametrized family $F=\{f_\eta\}_{\eta\in(-\delta,\delta)}$ of C^r diffeomorphisms of M for some $r\geqslant 1$ and $\delta>0$. Then, we call F a C^r *family* if the map

$$(-\delta,\delta)\ni\eta\mapsto f_\eta\in\mathrm{Diff}^r(M)$$

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is of class C^r. We assume that $f = f_0$ is an Axiom A surface diffeomorphism and that $\Lambda = \Lambda_0 \subset M$ is a basic set for f. (In this paper we always consider mixing basic sets to argue simply. (see Appendix A).) By structural stability of hyperbolic set, there exists $\delta_0 \in (0, \delta]$ such that for every $\eta \in (-\delta_0, \delta_0)$, f_η has a basic set $\Lambda_\eta \subset M$ and there exists a homeomorphism $h_\eta : \Lambda \to \Lambda_\eta$ that gives a topological conjugacy; that is $f_\eta \circ h_\eta = h_\eta \circ f$. See Lemma 3.1.

For each $\eta \in (-\delta_0, \delta_0)$ and for every $x \in \Lambda_\eta$, let us define the functions

$$\psi^s_\eta(x) = \log \|D_x f_\eta|_{E^s_\eta(x)}\|, \quad \psi^u_\eta(x) = -\log \|D_x f_\eta|_{E^u_\eta(x)}\|,$$

where E^s_η and E^u_η are the subbundles associated to the Df_η-invariant decomposition $T_{\Lambda_\eta} M = E^s_\eta \oplus E^u_\eta$. Put $\psi^s(x) = \psi^s_0(x)$ and $\psi^u(x) = \psi^u_0(x)$, respectively. Since these stable and unstable distributions $E^s_\eta(x)$ and $E^u_\eta(x)$ depend Hölder continuously on $x \in \Lambda_\eta$, so do the functions $\psi^s_\eta(x)$ and $\psi^u_\eta(x)$. Also let φ_η be a Hölder cotinuous potential on Λ_η. Then the numbers $T^s(\eta, q)$ and $T^u(\eta, q)$ are uniquely determined by the equations

$$P_\eta(T^s(\eta, q)\psi^s_\eta + q(\varphi_\eta - P_\eta(\varphi_\eta))) = 0$$

and

$$P_\eta(T^u(\eta, q)\psi^u_\eta + q(\varphi_\eta - P_\eta(\varphi_\eta))) = 0,$$

respectively, where $P_\eta(\cdot) = P_{\Lambda_\eta}(f_\eta, \cdot)$ denotes the topological pressure of f_η. (see Section 2 for its definition).

We write, for notational simplicity,

$$\begin{aligned} V^s(\eta, q) &= T^s(\eta, q)\psi^s_\eta + q(\varphi_\eta - P_\eta(\varphi_\eta)), \\ V^u(\eta, q) &= T^u(\eta, q)\psi^u_\eta + q(\varphi_\eta - P_\eta(\varphi_\eta)), \\ C_\mu(\varphi, \psi) &= \sum_{k \in \mathbb{Z}} \int \varphi \left(\psi \circ f^k - \int \psi \, d\mu \right) d\mu. \end{aligned}$$

Put $\delta^s = T^s(0, 0)$, $\delta^u = T^u(0, 0)$. Let μ^s and μ^u be the unique equilibrium states on Λ for $V^s(0, 0)$ and $V^u(0, 0)$, respectively.

Theorem 1.1. Let $F = \{f_\eta\}_{\eta \in (-\delta, \delta)}$ be a C^r family of $C^r (r \geqslant 3)$ diffeomorphisms of a compact surface M so that $f = f_0$ is an Axiom A diffeomorphism and that $\Lambda = \Lambda_0$ is a basic set for f. Then there exists

$\delta_1 \in (0, \delta_0]$ such that the Hausdorff dimension $\dim_H \Lambda_\eta$ is of class C^{r-2} in $\eta \in (-\delta_1, \delta_1)$. Furthermore

$$\frac{d^2}{d\eta^2}\bigg|_{\eta=0} \dim_H \Lambda_\eta = -\frac{1}{\{\int \psi^s d\mu^s\}^2}\delta^s \tilde{T}^s - \frac{1}{\{\int \psi^u d\mu^u\}^2}\delta^u \tilde{T}^u, \qquad (1.1)$$

where

$$\begin{aligned}
\tilde{T}^\tau &= A^\tau \int \psi^\tau d\mu^\tau - B^\tau \int \frac{d}{d\eta}\bigg|_{\eta=0} \psi_\eta^\tau \circ h_\eta \, d\mu^\tau - \left\{\int \frac{d}{d\eta}\bigg|_{\eta=0} \psi_\eta^\tau \circ h_\eta \, d\mu^\tau\right\}^2, \\
A^\tau &= \int \frac{d^2}{d\eta^2}\bigg|_{\eta=0} \psi_\eta^\tau \circ h_\eta \, d\mu^\tau + C_{\mu^\tau}\left(\frac{d}{d\eta}\bigg|_{\eta=0} \psi_\eta^\tau \circ h_\eta, \frac{d}{d\eta}\bigg|_{\eta=0} V^\tau(\eta, 0)\right), \\
B^\tau &= \int \frac{d}{d\eta}\bigg|_{\eta=0} \psi_\eta^\tau \circ h_\eta \, d\mu^\tau + C_{\mu^\tau}\left(\psi^\tau, \frac{d}{d\eta}\bigg|_{\eta=0} V^\tau(\eta, 0)\right), \quad (\tau = s, u).
\end{aligned}$$

Theorem 1.1 is a particular case of more general formula established in Section 4 where, indeed, given a parametrized family $\{\varphi_{\eta,\rho}\}_\rho$ of Hölder potentials on Λ_η we consider how do the associated numbers $T^s(\eta, \rho, q)$ and $T^u(\eta, \rho, q)$ vary with respect both to η and ρ. (See Sections 3 and 5 also). In ref. 1 Section 3.3, Barreira considered similar problem but the argument is not clear.

From (1.1) it is easy to see that if $\eta_0 \in (-\delta_1, \delta_1)$ is an inflection point of $\psi_\eta^\tau \circ h_\eta$, then, so is of $\dim_H \Lambda_\eta$.

Since $-\int \psi^\tau d\mu^\tau = h_{\mu^\tau}(f)$ $(\tau = s, u)$, the Eq. (1.1) can be rewrite as

$$\frac{d^2}{d\eta^2}\bigg|_{\eta=0} \dim_H \Lambda_\eta = -\frac{1}{h_{\mu^s}(f)^2}\delta^s \tilde{T}^s - \frac{1}{h_{\mu^u}(f)^2}\delta^u \tilde{T}^u,$$

where $h_\nu(f)$ denotes the measure theoretic entropy of f with respect to ν.

The paper is organaized as follows. Section 2 gives the definition and necessary properties of the topological pressure. Sections 3 and 4 are devoted to the proof of the results. In Section 5 the remaining variational formulae of the dimension spectra and also some identities on the spectra (Corollary 5.5) are stated. Section 6 gives necessary notions of the hyperbolic dynamical systems in short.

2. PROPERTIES OF PRESSURE

Let X be a compact metric space. We say that a function $\varphi : X \to \mathbb{R}$ is *Hölder continuous with exponents* α, or simply α-Hölder continuous,

$0 < \alpha \leqslant 1$, if

$$|\varphi(x) - \varphi(y)| \leqslant C d(x, y)^{\alpha} \quad (x, y \in X)$$

for some constant $C > 0$. Denote the set of all α-Hölder continuous functions on X by $C^{\alpha}(X)$. We shall consider $C^{\alpha}(X)$ endowed with the norm $\|\cdot\|_{\alpha}$ given by

$$\|\varphi\|_{\alpha} = \sup_{x} |\varphi(x)| + \sup_{x \neq y} \frac{|\varphi(x) - \varphi(y)|}{d(x, y)^{\alpha}}.$$

Also $C(X)$ denotes the space of continuous functions, and let $f : X \to X$ be a homeomorphism. We briefly recall several known facts of topological pressure. (See refs. 2 and 6 for more precise details).

In this paper the *topological pressure* $P_X(f, \cdot) : C(X) \to \mathbb{R}$ shall be defined via., the variational principle

$$P_X(f, \varphi) = \sup_{\mu} \left\{ h_{\mu}(f) + \int \varphi \, d\mu \right\},$$

where the supremum is taken over all f-invariant Borel probability measures on X.

(1) Let X_i be a compact metric space $f_i : X_i \to X_i$ a homeomorphism, $i = 1, 2$. Suppose $f_1 : X_1 \to X_1$ and $f_2 : X_2 \to X_2$ are topologically conjugate via., a homeomorphism $h : X_1 \to X_2$; that is, $h \circ f_1 = f_2 \circ h$. Then

$$P_{X_1}(f_1, \varphi \circ h) = P_{X_2}(f_2, \varphi), \quad \varphi \in C(X_2).$$

A measure μ is called *equilibrium state* for φ if $P_X(f, \varphi) = h_{\mu}(f) + \int \varphi \, d\mu$.

(2) Let Λ be a basic set for Axiom A diffeomorphism f of a compact manifold.

(i) Let φ a Hölder continuous function on Λ. Then there is a unique equilibrium state on Λ for φ which we denote by μ_{φ}.

(ii) For every $\alpha \in (0, 1)$, $P_X(f, \cdot) : C^{\alpha}(\Lambda) \to \mathbb{R}$ is real analytic function.

(iii) Let $\xi, \varphi, \psi \in C^{\alpha}(\Lambda)$, $\alpha \in (0, 1)$. Then

$$\left. \frac{\partial^2}{\partial \eta_1 \partial \eta_2} \right|_{\substack{\eta_1 = 0 \\ \eta_2 = 0}} P_X(f, \xi + \eta_1 \varphi + \eta_2 \psi) = C_{\mu_{\xi}}(\varphi, \psi).$$

(iv) For every $\alpha \in (0,1)$, $\mu_{\cdot} : C^{\alpha}(\Lambda) \to C^{\alpha}(\Lambda)^*$ is real analytic function.

3. PRELIMINARIES

Lemma 3.1. (ref. 3). If Λ is a basic set for $f \in \mathrm{Diff}^r(M)$, $r \geqslant 3$, then there exist $\delta_0 > 0$, $\alpha \in (0,1)$, and a C^{r-1} map

$$(-\delta_0, \delta_0) \ni \eta \mapsto h_\eta \in C^{\alpha}(\Lambda, M),$$

satisfying the following:

(1) $h_\eta : \Lambda \to \Lambda_\eta$ gives a topological conjugacy, where Λ_η is a basic set for f_η.

(2) The map $(-\delta_0, \delta_0) \ni \eta \mapsto \psi_\eta^\tau \circ h_\eta \in C^{\alpha}(\Lambda)$ is of class C^{r-2} ($\tau = s, u$).

For every $\eta \in (-\delta_0, \delta_0)$, let $\Phi_\eta = \{\varphi_{\eta,\rho}\}_{\rho \in (-\gamma,\gamma)}$ be a parametrized family of α-Hölder potentials on Λ_η for some $\gamma > 0$. We say that Φ_η is a C^k *family* if the map

$$(-\gamma, \gamma) \ni \rho \mapsto \varphi_{\eta,\rho} \in C^{\alpha}(\Lambda_\eta)$$

is of class C^k for $k \geqslant 1$.

For C^r family F and $\Phi = \{\Phi_\eta\}_\eta$, where each Φ_η is C^k family, we call the pair (F, Φ) a C^r *proper* if the map

$$(-\delta, \delta) \ni \eta \mapsto \varphi_{\eta,\rho} \in C^{\alpha}(M)$$

is of class C^r.

Put $\Psi_\eta^s = \psi_\eta^s \circ h_\eta$, and $\Psi_\eta^u = \psi_\eta^u \circ h_\eta$, respectively.

Remark 3.2. It follows from Lemma 3.1 that the pairs of perturbations (F, Ψ^s) and (F, Ψ^u) are C^{r-2} proper if Λ is a basic set for $f \in \mathrm{Diff}^r(M)$, $r \geqslant 3$.

Let ψ_η^s and ψ_η^u be the Hölder continuous functions defined in Section 1. Then the numbers $T^s(\eta,\rho,q)$ and $T^u(\eta,\rho,q)$ are uniquely determined by the equations

$$P_\eta(T^s(\eta,\rho,q)\psi_\eta^s + q(\varphi_{\eta,\rho} - P_\eta(\varphi_{\eta,\rho}))) = 0$$

and

$$P_\eta(T^u(\eta,\rho,q)\psi_\eta^u + q(\varphi_{\eta,\rho} - P_\eta(\varphi_{\eta,\rho}))) = 0,$$

respectively.

In what follows we write, for notational simplicity

$$\begin{aligned} V^s(\eta,\rho,q) &= T^s(\eta,\rho,q)\psi_\eta^s + q(\varphi_{\eta,\rho} - P_\eta(\varphi_{\eta,\rho})),\\ V^u(\eta,\rho,q) &= T^u(\eta,\rho,q)\psi_\eta^u + q(\varphi_{\eta,\rho} - P_\eta(\varphi_{\eta,\rho})),\\ \xi_\Lambda(\eta,\rho) &= \varphi_{\eta,\rho}\circ h_\eta - P(\varphi_{\eta,\rho}\circ h_\eta),\\ \xi_{\Lambda_\eta}(\eta,\rho) &= \varphi_{\eta,\rho} - P_\eta(\varphi_{\eta,\rho}),\\ \mu(g) &= \int g\,d\mu. \end{aligned}$$

Let $\mu_{0,\rho}^s$ and $\mu_{0,\rho}^u$ be the unique equilibrium states on Λ for $V^s(0,\rho,q)$ and $V^u(0,\rho,q)$, respectively. Next lemma describes the derivative formula for the perturbation of the dynamical system.

Lemma 3.3. Let $F=\{f_\eta\}_{\eta\in(-\delta,\delta)}$ be a C^r family of $C^r(r\geqslant 3)$ diffeomorphisms of a compact surface M so that $f=f_0$ is an Axiom A diffeomorphism and that $\Lambda=\Lambda_0$ is a basic set for f. Suppose $(-\delta_0,\delta_0)\ni\eta\mapsto\varphi_{\eta,\rho}$ is of class $C^k(k\geqslant r-2)$. Then there exists $\delta_1\in(0,\delta_0]$ such that T^s and T^u are of class C^{r-2} in $\eta\in(-\delta_1,\delta_1)$. Furthermore

$$\begin{aligned} &\frac{\partial}{\partial\eta}\Big|_{\eta=0} T^\tau(\eta,\rho,q)\\ &= -\frac{1}{\mu_{0,\rho}^\tau(\Psi_0^\tau)}\left\{T^\tau(0,\rho,q)\mu_{0,\rho}^\tau\left(\frac{d}{d\eta}\Big|_{\eta=0}\Psi_\eta^\tau\right) + q\mu_{0,\rho}^\tau\left(\frac{\partial}{\partial\eta}\Big|_{\eta=0}\xi_\Lambda(\eta,\rho)\right)\right\} \end{aligned}$$

for every $q\in\mathbb{R}$ $(\tau=s,u)$.

Let $\mu_{\eta,0}^s$ and $\mu_{\eta,0}^u$ be the unique equilibrium states on Λ_η for $V^s(\eta,0,q)$ and $V^u(\eta,0,q)$, respectively. Next lemma describes the derivative formula for the perturbation of the potential.

Lemma 3.4. Let Λ_η be a basic set for $f_\eta\in\mathrm{Diff}^r(M)$ and $\Phi_\eta=\{\varphi_{\eta,\rho}\}_{\rho\in(-\gamma,\gamma)}$ a $C^k(k\geqslant 1)$ family of Hölder continuous potentials on Λ_η for f_η. Then there exists $\gamma_1\in(0,\gamma]$ such that T^s and T^u are of class C^k in $\rho\in(-\gamma_1,\gamma_1)$. Furthermore

$$\frac{\partial}{\partial\rho}\Big|_{\rho=0} T^\tau(\eta,\rho,q) = -\frac{q}{\mu_{\eta,0}^\tau(\psi_\eta^\tau)}\mu_{\eta,0}^\tau\left(\frac{\partial}{\partial\rho}\Big|_{\rho=0}\xi_{\Lambda_\eta}(\eta,\rho)\right)$$

for every $q\in\mathbb{R}$ $(\tau=s,u)$.

These two lemmata can be easily deduced from the argument in refs. 1 and 8 with slight modification and hence we omit the proofs.

Lemma 3.5. Let X be a compact metric space and $\xi, \varphi \in C^\alpha(X)$. Suppose $t \mapsto \mu_t$ is continuous. Then

$$\frac{d}{d\eta}\bigg|_{\eta=0} \mu_{\xi+\eta\varphi}(\xi + \eta\varphi) = \mu_\xi(\varphi) + C_{\mu_\xi}(\xi, \varphi).$$

Proof. Let us denote

$$I(\xi, \varphi) = \mu_\xi(\varphi) + C_{\mu_\xi}(\xi, \varphi).$$

Clearly

$$\frac{\mu_\xi(\xi + \eta\varphi) - \mu_\xi(\xi)}{\eta} = \mu_\xi(\varphi).$$

We know

$$\frac{\mu_{\xi+\eta\varphi}(\xi) - \mu_\xi(\xi)}{\eta} = C_{\mu_\xi}(\xi, \varphi) + o(\eta),$$

as $\eta \to 0$. (see ref. 6; chapter 7 or ref. 7; Section 3). Henceforce

$$\begin{aligned}
&\left| \frac{\mu_{\xi+\eta\varphi}(\xi + \eta\varphi) - \mu_\xi(\xi)}{\eta} - I(\xi, \varphi) \right| \\
&\quad = \left| \frac{\mu_{\xi+\eta\varphi}(\xi + \eta\varphi) - \mu_\xi(\xi + \eta\varphi)}{\eta} + \frac{\mu_\xi(\xi + \eta\varphi) - \mu_\xi(\xi)}{\eta} - I(\xi, \varphi) \right| \\
&\quad \leqslant \left| \frac{\mu_{\xi+\eta\varphi}(\xi) - \mu_\xi(\xi)}{\eta} - C_{\mu_\xi}(\xi, \varphi) \right| + \left| \frac{\mu_{\xi+\eta\varphi}(\eta\varphi) - \mu_\xi(\eta\varphi)}{\eta} \right| \\
&\quad \leqslant |\eta| \|\varphi\|^2 + o(\eta),
\end{aligned}$$

as $\eta \to 0$. Thereby, the lemma proved. ∎

Lemma 3.6. Let X be a compact metric space and $\xi, \varphi_\eta \in C^\alpha(X)$. Suppose $\eta \mapsto \varphi_\eta$ and $t \mapsto \mu_t$ are both continuous. Then,

$$\frac{d}{d\eta}\bigg|_{\eta=0} \mu_{\xi+\eta\varphi_\eta}(\xi + \eta\varphi_\eta) = \mu_\xi(\varphi_0) + C_{\mu_\xi}(\xi, \varphi_0).$$

Proof. Let

$$I(\xi,\varphi_0)=\mu_\xi(\varphi_0)+C_{\mu_\xi}(\xi,\varphi_0).$$

Then, by the previous lemma

$$\left|\frac{\mu_{\xi+\eta\varphi_0}(\xi+\eta\varphi_0)-\mu_\xi(\xi)}{\eta}-I(\xi,\varphi_0)\right|\leqslant o(\eta),$$

as $\eta\to 0$. Furthermore,

$$\begin{aligned}
&\left|\frac{\mu_{\xi+\eta\varphi_\eta}(\xi+\eta\varphi_\eta)-\mu_{\xi+\eta\varphi_0}(\xi+\eta\varphi_0)}{\eta}\right|\\
&\leqslant\left|\frac{\mu_{\xi+\eta\varphi_\eta}(\xi+\eta\varphi_\eta)-\mu_{\xi+\eta\varphi_\eta}(\xi+\eta\varphi_0)}{\eta}\right|\\
&\quad+\left|\frac{\mu_{\xi+\eta\varphi_\eta}(\xi+\eta\varphi_0)-\mu_{\xi+\eta\varphi_0}(\xi+\eta\varphi_0)}{\eta}\right|\\
&\leqslant\|\varphi_\eta-\varphi_0\|+\|\xi+\eta\varphi_0\|\cdot\|\varphi_\eta-\varphi_0\|.
\end{aligned}$$

In view of the considerations above

$$\begin{aligned}
&\left|\frac{\mu_{\xi+\eta\varphi_\eta}(\xi+\eta\varphi_\eta)-\mu_\xi(\xi)}{\eta}-I(\xi,\varphi_0)\right|\\
&\leqslant\left|\frac{\mu_{\xi+\eta\varphi_\eta}(\xi+\eta\varphi_\eta)-\mu_{\xi+\eta\varphi_0}(\xi+\eta\varphi_0)}{\eta}\right|\\
&\quad+\left|\frac{\mu_{\xi+\eta\varphi_0}(\xi+\eta\varphi_0)-\mu_\xi(\xi)}{\eta}-I(\xi,\varphi_0)\right|\\
&\leqslant\|\varphi_\eta-\varphi_0\|+\|\xi+\eta\varphi_0\|\cdot\|\varphi_\eta-\varphi_0\|+o(\eta)\\
&\leqslant|\eta|(1+\|\xi+\eta\varphi_0\|)+o(\eta)
\end{aligned}$$

as $\eta\to 0$, thereby completing the proof of Lemma 3.6. ∎

4. PROOFS

Proposition 4.1. Suppose that the same assumption of Lemma 3.3 holds for $r\geqslant 4$. Then there exists $\delta_1\in(0,\delta_0]$ such that T^s and T^u are of class C^{r-2} in $\eta\in(-\delta_1,\delta_1)$. Furthermore

$$\frac{\partial^2}{\partial\eta^2}\Big|_{\eta=0} T^\tau(\eta,\rho,q) = -\frac{q}{\mu^\tau_{0,\rho}(\psi^\tau)^2}\{\mu^\tau_{0,\rho}(\psi^\tau)\tilde{T}^\tau_1(\rho,q) - \tilde{T}^\tau_2(\rho,q)\}$$
$$-\frac{T^\tau(0,\rho,q)}{\mu^\tau_{0,\rho}(\psi^\tau)^2}\tilde{T}^\tau_3(\rho,q),$$

where

$$\begin{aligned}
\tilde{T}^\tau_1(\rho,q) &= \mu^\tau_{0,\rho}\left(\frac{\partial^2}{\partial\eta^2}\Big|_{\eta=0}\xi_\Lambda(\eta,\rho)\right)\\
&\quad + C_{\mu_{0,\rho}}\left(\frac{\partial}{\partial\eta}\Big|_{\eta=0}\xi_\Lambda(\eta,\rho), \frac{\partial}{\partial\eta}\Big|_{\eta=0} V^\tau(\eta,\rho,q)\right),\\
\tilde{T}^\tau_2(\rho,q) &= \mu^\tau_{0,\rho}\left(\frac{\partial}{\partial\eta}\Big|_{\eta=0}\xi_\Lambda(\eta,\rho)\right)\left\{\mu^\tau_{0,\rho}\left(\frac{d}{d\eta}\Big|_{\eta=0}\Psi^\tau_\eta\right) + B^\tau(\rho,q)\right\},\\
\tilde{T}^\tau_3(\rho,q) &= A^\tau(\rho,q)\mu^\tau_{0,\rho}(\psi^\tau) - B^\tau(\rho,q)\mu^\tau_{0,\rho}\left(\frac{d}{d\eta}\Big|_{\eta=0}\Psi^\tau_\eta\right)\\
&\quad -\mu^\tau_{0,\rho}\left(\frac{d}{d\eta}\Big|_{\eta=0}\Psi^\tau_\eta\right)^2,\\
A^\tau(\rho,q) &= \mu^\tau_{0,\rho}\left(\frac{d^2}{d\eta^2}\Big|_{\eta=0}\Psi^\tau_\eta\right) + C_{\mu^\tau_{0,\rho}}\left(\frac{d}{d\eta}\Big|_{\eta=0}\Psi^\tau_\eta, \frac{\partial}{\partial\eta}\Big|_{\eta=0}V^\tau(\eta,\rho,q)\right),\\
B^\tau(\rho,q) &= \mu^\tau_{0,\rho}\left(\frac{d}{d\eta}\Big|_{\eta=0}\Psi^\tau_\eta\right) + C_{\mu^\tau_{0,\rho}}\left(\psi^\tau, \frac{\partial}{\partial\eta}\Big|_{\eta=0}V^\tau(\eta,\rho,q)\right)
\end{aligned}$$

for every $q\in\mathbb{R}$ $(\tau = s, u)$.

Proof. It follows from Lemmas 3.3 and 3.6 that:

$$\begin{aligned}
&-\mu^\tau_{0,\rho}(\Psi^\tau_0)^2\frac{\partial^2}{\partial\eta^2}\Big|_{\eta=0}T^\tau(\eta,\rho,q)\\
&= \left\{\frac{\partial}{\partial\eta}\Big|_{\eta=0}T^\tau(\eta,\rho,q)\mu^\tau_{0,\rho}\left(\frac{d}{d\eta}\Big|_{\eta=0}\Psi^\tau_\eta\right)\right.\\
&\quad + T^\tau(0,\rho,q)\left\{\mu^\tau_{0,\rho}\left(\frac{d^2}{d\eta^2}\Big|_{\eta=0}\Psi^\tau_\eta\right)\right.\\
&\quad \left.+ C_{\mu^\tau_{0,\rho}}\left(\frac{d}{d\eta}\Big|_{\eta=0}\Psi^\tau_\eta, \frac{\partial}{\partial\eta}\Big|_{\eta=0}V^\tau(\eta,\rho,q)\right)\right\}\\
&\quad + q\left(\mu^\tau_{0,\rho}\left(\frac{\partial^2}{\partial\eta^2}\Big|_{\eta=0}\xi_\Lambda(\eta,\rho)\right)\right.
\end{aligned}$$

$$
\begin{aligned}
&+C_{\mu^\tau_{0,\rho}}\left(\frac{\partial}{\partial\eta}\Big|_{\eta=0}\xi_\Lambda(\eta,\rho),\frac{\partial}{\partial\eta}\Big|_{\eta=0}V^\tau(\eta,\rho,q)\right)\Bigg)\Bigg\}\\
&\times\mu^\tau_{0,\rho}(\Psi^\tau_0)-\left\{T^\tau(0,\rho,q)\mu^\tau_{0,\rho}\left(\frac{d}{d\eta}\Big|_{\eta=0}\Psi^\tau_\eta\right)+q\mu^\tau_{0,\rho}\left(\frac{\partial}{\partial\eta}\Big|_{\eta=0}\xi_\Lambda(\eta,\rho)\right)\right\}\\
&\times\left\{\mu^\tau_{0,\rho}\left(\frac{d}{d\eta}\Big|_{\eta=0}\Psi^\tau_\eta\right)+C_{\mu^\tau_{0,\rho}}\left(\Psi^\tau_0,\frac{\partial}{\partial\eta}\Big|_{\eta=0}V^\tau(\eta,\rho,q)\right)\right\}.
\end{aligned}
$$

Applying Lemma 3.3 again will complete the proof of Proposition 4.1 (note $\Psi^\tau_0=\psi^\tau_0=\psi^\tau$). ∎

Proposition 4.2. Suppose that the same assumption of Lemma 3.4 holds for $k\geqslant 2$. Then there exists $\gamma_1\in(0,\gamma]$ such that T^s and T^u are of class C^k in $\rho\in(-\gamma_1,\gamma_1)$. Furthermore

$$
\frac{\partial^2}{\partial\rho^2}\Big|_{\rho=0}T^\tau(\eta,\rho,q)=-\frac{q}{\mu^\tau_{\eta,0}(\psi^\tau_\eta)^2}\{\mu^\tau_{\eta,0}(\psi^\tau_\eta)\tilde{T}^\tau_4(\eta,q)-\tilde{T}^\tau_5(\eta,q)\},
$$

where

$$
\begin{aligned}
\tilde{T}^\tau_4(\eta,q)&=\mu^\tau_{\eta,0}\left(\frac{\partial^2}{\partial\rho^2}\Big|_{\rho=0}\xi_{\Lambda_\eta}(\eta,\rho)\right)\\
&\qquad+C_{\mu^\tau_{\eta,0}}\left(\frac{\partial}{\partial\rho}\Big|_{\rho=0}\xi_{\Lambda_\eta}(\eta,\rho),\frac{\partial}{\partial\rho}\Big|_{\rho=0}V^\tau(\eta,\rho,q)\right),\\
\tilde{T}^\tau_5(\eta,q)&=\mu^\tau_{\eta,0}\left(\frac{\partial}{\partial\rho}\Big|_{\rho=0}\xi_{\Lambda_\eta}(\eta,\rho)\right)C_{\mu^\tau_{\eta,0}}\left(\psi^\tau_\eta,\frac{\partial}{\partial\rho}\Big|_{\rho=0}V^\tau(\eta,\rho,q)\right)
\end{aligned}
$$

for every $q\in\mathbb{R}$ $(\tau=s,u)$.

Proof. It follows from Lemmas 3.4 and 3.6 that

$$
\begin{aligned}
&-\frac{\mu^\tau_{\eta,0}(\psi^\tau_\eta)^2}{q}\frac{\partial^2}{\partial\rho^2}\Big|_{\rho=0}T^\tau(\eta,\rho,q)\\
&=\left\{\mu^\tau_{\eta,0}\left(\frac{\partial^2}{\partial\rho^2}\Big|_{\rho=0}\xi_{\Lambda_\eta}(\eta,\rho)\right)\right.\\
&\left.+C_{\mu^\tau_{\eta,0}}\left(\frac{\partial}{\partial\rho}\Big|_{\rho=0}\xi_{\Lambda_\eta}(\eta,\rho),\frac{\partial}{\partial\rho}\Big|_{\rho=0}V^\tau(\eta,\rho,q)\right)\right\}\\
&\times\mu^\tau_{\eta,0}(\psi^\tau_\eta)-\mu^\tau_{\eta,0}\left(\frac{\partial}{\partial\rho}\Big|_{\rho=0}\xi_{\Lambda_\eta}(\eta,\rho)\right)C_{\mu^\tau_{\eta,0}}\left(\psi^\tau_\eta,\frac{\partial}{\partial\rho}\Big|_{\rho=0}V^\tau(\eta,\rho,q)\right).
\end{aligned}
$$

Thus Proposition 4.2 is obtained. ∎

Denote the local stable and unstable manifolds for f_η by $\mathcal{W}^s_\eta(x)$ and $\mathcal{W}^u_\eta(x)$, respectively. Put $\mathcal{W}^s(x)=\mathcal{W}^s_0(x)$ and $\mathcal{W}^u(x)=\mathcal{W}^u_0(x)$. (see Section Appendix A for its definitions).

Proposition 4.3. (refs. 4 and 5) Let f be a C^1 Axiom A surface diffeomorphism. Then the following hold:

(i) $\dim_H \mathcal{W}^s(x)\cap\Lambda=\delta^s, \quad \dim_H \mathcal{W}^u(x)\cap\Lambda=\delta^u \quad (x\in\Lambda)$,

(ii) $\dim_H \Lambda=\delta^s+\delta^u$.

Proposition 4.4. (refs. 3 and 5) Let $F=\{f_\eta\}_{\eta\in(-\delta,\delta)}$ be a $C^r(r\geqslant 3)$ family of C^r diffeomorphisms of a compact surface M so that $f=f_0$ is an Axiom A diffeomorphism and that $\Lambda=\Lambda_0$ is a basic set for f. Then the functions

$$(-\delta_1,\delta_1)\ni\eta\mapsto\dim_H \mathcal{W}^s_\eta(x)\cap\Lambda_\eta,$$
$$(-\delta_1,\delta_1)\ni\eta\mapsto\dim_H \mathcal{W}^u_\eta(x)\cap\Lambda_\eta$$

are independent of $x\in\Lambda$ and are of class C^{r-2}.

Proof of Theorem 1.1. By setting $q=0$ in Proposition 4.1 and combining Propositions 4.3 and 4.4 we obtain Theorem 4.1. ▮

5. OTHER VARIATIONAL FORMULAE

Assumption (A). Let $F=\{f_\eta\}_{\eta\in(-\delta,\delta)}$ be a $C^r(r\geqslant 3)$ family of C^r diffeomorphisms of a compact surface M so that $f=f_0$ is an Axiom A diffeomorphism and that $\Lambda=\Lambda_0$ is a basic set for f, and $\Phi_\eta=\{\varphi_{\eta,\rho}\}_{\rho\in(-\gamma,\gamma)}$ a $C^k(k\geqslant 1)$ family of Hölder continuous potentials on the basic set Λ_η for f_η, $\eta\in(-\delta_0,\delta_0)$ for some $\delta_0\in(0,\delta]$. Suppose the pair $(F,\{\Phi_\eta\}_\eta)$ is C^r proper.

Theorem 5.1. Suppose that the Assumption (A) holds. Then there exist $\delta_1\in(0,\delta_0]$, and $\gamma_1\in(0,\gamma]$ such that the functions T^s and T^u are of class C^{r-2} in $\eta\in(-\delta_1,\delta_1)$, and that are of class C^k in $\rho\in(-\gamma_1,\gamma_1)$, and that are real analytic in $q\in\mathbb{R}$.

Proof. Obviously Theorem 5.1 can be proved by each first ststement of Lemmata 3.3 and 3.4. ▮

Let μ^s_q and μ^u_q be the equilibrium states on Λ for $V^s(0,0,q)$ and $V^u(0,0,q)$, respectively. Furthermore the following formulae hold.

Proposition 5.2. Let $\tau = s, u$. Suppose that the assumption (A) holds. Then we have the following:

$$\begin{aligned}\frac{\partial^2}{\partial\eta\partial\rho}\Big|_{\substack{\eta=0\\ \rho=0}} T^\tau(\eta,\rho,q) &= \frac{\partial}{\partial\rho}\Big|_{\rho=0}\left(\frac{\partial}{\partial\eta}\Big|_{\eta=0} T^\tau(\eta,\rho,q)\right)\\ &= -\frac{q}{\mu_q^\tau(\psi^\tau)^2}\{\mu_q^\tau(\psi^\tau)T_1^\tau(q) - T_2^\tau(q)\}\\ &\quad -\frac{T^\tau(0,0,q)}{\mu_q^\tau(\psi^\tau)^2}T_3^\tau(q),\end{aligned} \tag{5.1}$$

where

$$\begin{aligned}T_1^\tau(q) &= \mu_q^\tau\left(\frac{\partial^2}{\partial\eta\partial\rho}\Big|_{\substack{\eta=0\\ \rho=0}}\xi_\Lambda(\eta,\rho)\right)\\ &\quad + C_{\mu_q^\tau}\left(\frac{\partial}{\partial\eta}\Big|_{\eta=0}\xi_\Lambda(\eta,0), \frac{\partial}{\partial\rho}\Big|_{\rho=0}V^\tau(0,\rho,q)\right),\\ T_2^\tau(q) &= \mu_q^\tau\left(\frac{\partial}{\partial\rho}\Big|_{\rho=0}\xi_\Lambda(0,\rho)\right)\mu_q^\tau\left(\frac{d}{d\eta}\Big|_{\eta=0}\psi_\eta^\tau\circ h_\eta\right)\\ &\quad + \mu_q^\tau\left(\frac{\partial}{\partial\eta}\Big|_{\eta=0}\xi_\Lambda(\eta,0)\right)C_{\mu_q^\tau}\left(\psi^\tau, \frac{\partial}{\partial\rho}\Big|_{\rho=0}V^\tau(0,\rho,q)\right),\\ T_3^\tau(q) &= \mu_q^\tau(\psi^\tau)C_{\mu_0^\tau}\left(\frac{d}{d\eta}\Big|_{\eta=0}\psi_\eta^\tau\circ h_\eta, \frac{\partial}{\partial\rho}\Big|_{\rho=0}V^\tau(0,\rho,q)\right)\\ &\quad - \mu_q^\tau\left(\frac{d}{d\eta}\Big|_{\eta=0}\psi_\eta^\tau\circ h_\eta\right)C_{\mu_q^\tau}\left(\psi^\tau, \frac{\partial}{\partial\rho}\Big|_{\rho=0}V^\tau(0,\rho,q)\right)\end{aligned}$$

for every $q \in \mathbb{R}$.

Proof. It follows from Lemmas 3.3 and 3.6 that:

$$\begin{aligned}&-\mu_q^\tau(\Psi_0^\tau)^2\frac{\partial}{\partial\rho}\Big|_{\rho=0}\left(\frac{\partial}{\partial\eta}\Big|_{\eta=0}T^\tau(\eta,\rho,q)\right)\\ &= \Bigg\{\frac{\partial}{\partial\rho}\Big|_{\rho=0}T^\tau(0,\rho,q)\mu_q^\tau\left(\frac{d}{d\eta}\Big|_{\eta=0}\Psi_\eta^\tau\right)\\ &\quad + T^\tau(0,0,q)C_{\mu_q^\tau}\left(\frac{d}{d\eta}\Big|_{\eta=0}\Psi_\eta^\tau, \frac{\partial}{\partial\rho}\Big|_{\rho=0}V^\tau(0,\rho,q)\right)\\ &\quad + q\Bigg(\mu_q^\tau\left(\frac{\partial^2}{\partial\eta\partial\rho}\Big|_{\substack{\eta=0\\ \rho=0}}\xi_\Lambda(\eta,\rho)\right)\\ &\quad + C_{\mu_q^\tau}\left(\frac{\partial}{\partial\eta}\Big|_{\eta=0}\xi_\Lambda(\eta,0), \frac{\partial}{\partial\rho}\Big|_{\rho=0}V^\tau(0,\rho,q)\right)\Bigg)\Bigg\}\end{aligned}$$

$$\times\mu_q^\tau(\Psi_0^\tau)-\left\{T^\tau(0,0,q)\mu_q^\tau\left(\frac{d}{d\eta}\Big|_{\eta=0}\Psi_\eta^\tau\right)+q\mu_q^\tau\left(\frac{\partial}{\partial\eta}\Big|_{\eta=0}\xi_\Lambda(\eta,0)\right)\right\}$$
$$\times C_{\mu_q^\tau}\left(\Psi_0^\tau,\frac{\partial}{\partial\rho}\Big|_{\rho=0}V^\tau(0,\rho,q)\right).$$

Combining Lemma 3.4 for $\eta=0$ will complete the proof of Proposition 5.2 (note $\Psi_0^\tau=\psi_0^\tau=\psi^\tau$). ∎

Proposition 5.3. Let $\tau=s,u$. Suppose that the Assumption (A) holds. Then we have the following:

$$\begin{aligned}\frac{\partial^2}{\partial\rho\partial\eta}\Big|_{\substack{\rho=0\\ \eta=0}}T^\tau(\eta,\rho,q)&=\frac{\partial}{\partial\eta}\Big|_{\eta=0}\left(\frac{\partial}{\partial\rho}\Big|_{\rho=0}T^\tau(\eta,\rho,q)\right)\\&=-\frac{q}{\mu_q^\tau(\psi^\tau)^2}\{\mu_q^\tau(\psi^\tau)T_4^\tau(q)-T_5^\tau(q)\},\end{aligned}\tag{5.2}$$

where

$$\begin{aligned}T_4^\tau(q)&=\mu_q^\tau\left(\frac{\partial^2}{\partial\rho\partial\eta}\Big|_{\substack{\rho=0\\ \eta=0}}\xi_{\Lambda_\eta}(\eta,\rho)\right)\\&\quad+C_{\mu_q^\tau}\left(\frac{\partial}{\partial\rho}\Big|_{\rho=0}\xi_\Lambda(0,\rho),\frac{\partial}{\partial\eta}\Big|_{\eta=0}V^\tau(\eta,0,q)\right),\\T_5^\tau(q)&=\mu_q^\tau\left(\frac{\partial}{\partial\rho}\Big|_{\rho=0}\xi_\Lambda(0,\rho)\right)\left\{\mu_q^\tau\left(\frac{d}{d\eta}\Big|_{\eta=0}\psi_\eta^\tau\right)\right.\\&\quad\left.+C_{\mu_q^\tau}\left(\psi_0^\tau,\frac{\partial}{\partial\eta}\Big|_{\eta=0}V^\tau(\eta,0,q)\right)\right\}\end{aligned}$$

for every $q\in\mathbb{R}$.

Proof. Using Lemmas 3.4 and 3.6 shall imply that

$$\begin{aligned}&-\frac{\mu_q^\tau(\psi_0^\tau)^2}{q}\frac{\partial}{\partial\eta}\Big|_{\eta=0}\left(\frac{\partial}{\partial\rho}\Big|_{\rho=0}T^\tau(\eta,\rho,q)\right)\\&=\left\{\mu_q^\tau\left(\frac{\partial^2}{\partial\rho\partial\eta}\Big|_{\substack{\rho=0\\ \eta=0}}\xi_{\Lambda_\eta}(\eta,\rho)\right)\right.\\&\quad\left.+C_{\mu_q^\tau}\left(\frac{\partial}{\partial\rho}\Big|_{\rho=0}\xi_{\Lambda_0}(0,\rho),\frac{\partial}{\partial\eta}\Big|_{\eta=0}V^\tau(\eta,0,q)\right)\right\}\mu_q^\tau(\psi_0^\tau)\\&\quad-\mu_q^\tau\left(\frac{\partial}{\partial\rho}\Big|_{\rho=0}\xi_{\Lambda_0}(0,\rho)\right)\left\{\mu_q^\tau\left(\frac{d}{d\eta}\Big|_{\eta=0}\psi_\eta^\tau\right)\right.\\&\quad\left.+C_{\mu_q^\tau}\left(\psi_0^\tau,\frac{\partial}{\partial\eta}\Big|_{\eta=0}V^\tau(\eta,0,q)\right)\right\}.\end{aligned}$$

Thus Proposition 5.3 is obtained (note $\xi_{\Lambda_0}(0,\rho)=\xi_\Lambda(0,\rho)$). ∎

Remark 5.4. If $r\geqslant 4$ and $k\geqslant 2$, then the derivatives (5.1) and (5.2) are coincide, respectively, for every $q\in\mathbb{R}$.

Suppose $r\geqslant 4$ and $k\geqslant 2$. Then by Remark 5.4 setting $q=0$, in particular, implies

$$\delta^\tau T_3^\tau(0)=0 \quad (\tau=s,u),$$

since $\mu_0^\tau(\psi^\tau)\neq 0$ (note that $T^\tau(0,0,0)$ coincides δ^τ defined in Section 1). Hence (i) $T_3^s(0)=T_3^u(0)=0$, or (ii) $\delta^s=\delta^u=0$ might occur. By using the Young formula,[9] we can easily conclude that $\delta^s=\delta^u=0$ if and only if $h_{\mu_0^s}(f)=h_{\mu_0^u}(f)=0$. We know, however, that $h_{\mu_0^\tau}(f)=-\mu_0^\tau(\psi^\tau)\neq 0$, and hence case (ii) does not occur.

Next setting $q=1$ implies

$$\mu_1^\tau(\psi^\tau)\{T_1^\tau(1)-T_4^\tau(1)\}=T_2^\tau(1)-T_5^\tau(1) \quad (\tau=s,u),$$

since $T^\tau(0,0,1)=0$.

Hence we summarize as follows.

Corollary 5.5. Let $\tau=s,u$. Suppose that the Assumption (A) holds for $r\geqslant 4$ and $k\geqslant 2$. Then we have $T_3^\tau(0)=0$. Furthermore following (1), or (2) hold.

(1) $T_1^\tau(1)=T_4^\tau(1)$ (equivalently $T_2^\tau(1)=T_5^\tau(1)$)

(2) $h_{\mu_1^\tau}(f)=-\{T_2^\tau(1)-T_5^\tau(1)\}/\{T_1^\tau(1)-T_4^\tau(1)\}$.

APPENDIX A

Let $f:M\to M$ be a $C^{1+\alpha}(\alpha>0)$ diffeomorphism. A compact f-invariant set $\Lambda\subset M$ is said to be *hyperbolic* if there exists2 a continuous splitting of the tangent bundle $T_\Lambda M=E^s\oplus E^u$ and constants $C>0$ and $\lambda\in(0,1)$ such that for every $x\in\Lambda$,

(a) $D_xf(E^s(x))=E^s(fx)$ and $D_xf(E^u(x))=E^u(fx)$,

(b) for every $n\geqslant 0$, we have for all $v\in E^s(x)$

$$\|D_xf^n(v)\|\leqslant C\lambda^n\|v\|$$

for all $v \in E^u(x)$

$$\|D_x f^{-n}(v)\| \leqslant C\lambda^n \|v\|$$

The subspaces $E^s(x)$ and $E^u(x)$ depend Hölder continuously on $x \in \Lambda$.

Now let Λ be a hyperbolic set for f. If there exists an open neightborhood O of Λ such that

$$\Lambda = \bigcap_{i \in \mathbb{Z}} f^i(\bar{O}),$$

then we say $\Lambda = \Lambda_f$ a *basic* set or *locally maximal* set for f. A diffeomorphism $f : M \to M$ is said be *Axiom A* if its non-wandering set $\Omega(f)$ is a locally maximal hyperbolic set.

Let $x \in \Omega(f)$. We define the *local stable* and *unstable* manifolds for f as

$$\begin{aligned} \mathcal{W}^s(x) &= \{y \in M; d(f^n x, f^n y) \leqslant \varepsilon \quad (n \geqslant 0)\}, \\ \mathcal{W}^u(x) &= \{y \in M; d(f^{-n} x, f^{-n} y) \leqslant \varepsilon \quad (n \geqslant 0)\} \end{aligned}$$

for some $\varepsilon > 0$. Then they satisfy

$$T_x \mathcal{W}^s(x) = E^s(x), \quad T_x \mathcal{W}^s(x) = E^s(x).$$

By the Smale spectral decomposition theorem, the set $\Omega(f)$ can be decomposed into finitely many pairwise disjoint closed f-invariant locally maximal hyperbolic sets, say Ω_i, such that $f|_{\Omega_i} : \Omega_i \to \Omega_i$ is topologically transitive. Furthermore, for each i there exists a integer n_i and pairwise disjoint closed sets $\Lambda_{i,j}$ such that $f(\Lambda_{i,j}) = \Lambda_{i,j+1}$ $(\Lambda_{i,n_i+1} = \Lambda_{1,i})$ and $f^{n_i}|_{\Lambda_{i,j}} : \Lambda_{i,j} \to \Lambda_{i,j}$ is topologically mixing. We refer the reader to refs. 2 and 6 for more precise details.

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